

Minnesota State High School Mathematics League

Individual Event

2003-04 Event 5A

Questions in this event are written, with permission, as variations of problems from the 1995 AHSME. Review of this exam is excellent preparation, not only for this event, but for the 1996 AHSME, which we strongly encourage you to take.

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

3

1. How many non-congruent triangles with perimeter 9 have sides whose lengths are integers?

$\left. \begin{matrix} 4, 4, 1 \\ 4, 3, 2 \\ 3, 3, 3 \end{matrix} \right\}$ Three

6

2. A $4 \times 4 \times 4$ cube of cheese can be cut into 64 $1 \times 1 \times 1$ cubes by making nine cuts, but the same result (64 $1 \times 1 \times 1$ cubes) can be achieved using fewer cuts. What is the minimum number of cuts with which the desired result can be achieved?

Mon

Tues

Wed

3. The grid in Figure 3 is torn from a calendar for January, 2004. The dates in the marked squares sum to 80. What three days of the week are represented in the part of the calendar shown? (In case you forgot, New Year's Day was Thursday.)

17

4. Four people come to the north end of a rickety bridge that will support at most two people at a time. It is dark, so two people who are crossing must use a flashlight, and since only one light is available, it must be walked back to others waiting to cross. Albert can run across in 1 minute, but his sister Barbara requires 2 minutes. His mother, Cindy, requires 5 minutes, and his Grandpa Dotin requires 10 minutes. Naturally, two people crossing together must go at the speed of the slow person. What is the shortest amount of time in which the entire group can reach the south end of the bridge?

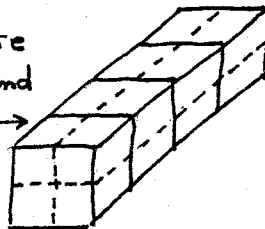
Time 0 2 3 5 10

South End
AB
BC
CD
ABCD

North End
ABCD
CD
ACD
AB

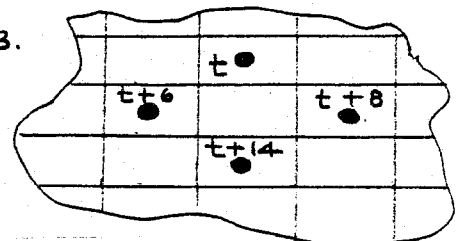
Trip
stay 1 2 3 4 5

2. Use 3 cuts to create four $2 \times 2 \times 2$ cubes, and arrange as indicated →
Two cuts (dotted lines) will give sixteen



$1 \times 1 \times 2$ blocks. Stack these blocks like dominoes. One more cut (the sixth) will finish the job.

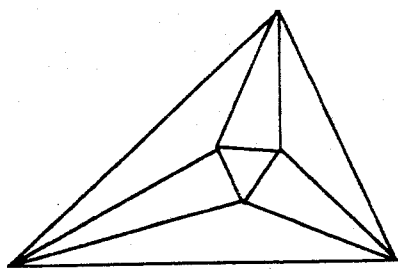
3.



$$t + (t+6) + (t+8) + (t+14) = 80$$

$$4t = 52; t = 13; 1, 8, 15, \dots$$

are Thursdays; 13 is Tuesday



Minnesota State High School Mathematics League Individual Event

2003-04 Event 5B

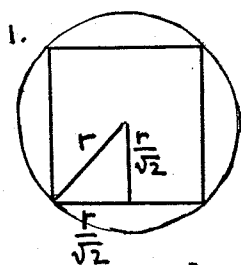
The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

2r² 1. A square is inscribed in a circle of radius r . Express the area of the square in terms of r .

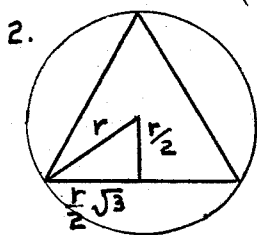
$\frac{3\sqrt{3}}{4} r^2$ 2. An equilateral triangle is inscribed in a circle of radius r . Express the area of the triangle in terms of r .

$\frac{9}{8}$ 3. A square and a regular hexagon have the same perimeter. Let S be the area of the circle circumscribing the square, and H be the area of the circle circumscribing the hexagon. Find $\frac{S}{H}$.

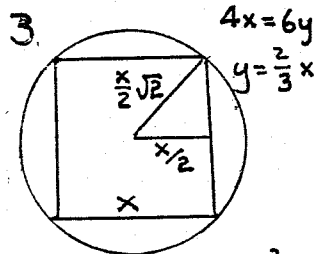
$\frac{4}{9}$ 4. Let $ABCD$ be a parallelogram with diagonals of $AC = 9$, $BD = 6$. Let the circle through A , B , and D intersect AC at E , and let the diagonals of the parallelogram meet at H . Find the ratio $\frac{\text{Area}(\triangle DHE)}{\text{Area}(\triangle ADH)}$.



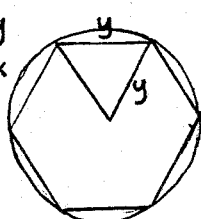
$$A = \left(2 \frac{r}{\sqrt{2}}\right)^2 = 2r^2$$



$$A = 3 \left[\frac{1}{2} r \sqrt{3} \frac{r}{2} \right] = \frac{3\sqrt{3}}{4} r^2$$



$$S = \pi \left(\frac{x}{\sqrt{2}} \right)^2 = \frac{\pi x^2}{2}$$



$$H = \pi y^2$$

$$\frac{S}{H} = \frac{\frac{\pi x^2}{2}}{\pi y^2} = \frac{x^2}{2 \left(\frac{2}{3} x \right)^2} = \frac{9}{8}$$

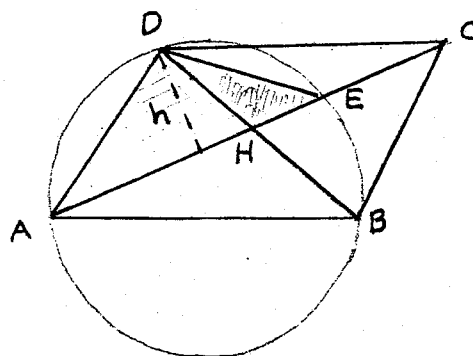
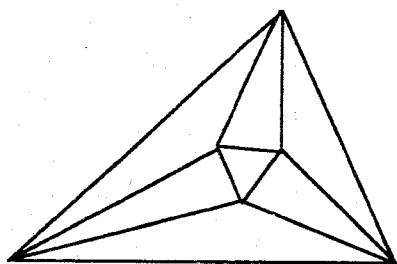


Figure 4

4. Since AE and DB are intersecting chords, $(AH)(HE) = (DH)(HB)$
 $\therefore HE = \frac{(DH)(HB)}{(AH)} = \frac{(\frac{1}{2} DB)(\frac{1}{2} DB)}{AH}$

Both $\triangle DHE$ and $\triangle ADH$ have altitude h

$$\frac{\text{Area}(\triangle DHE)}{\text{Area}(\triangle ADH)} = \frac{\frac{1}{2} h (HE)}{\frac{1}{2} h (AH)} = \frac{\frac{1}{4} (DB)^2}{(AH)^2} = \frac{(DB)^2}{(2AH)^2} = \frac{6^2}{9^2} = \frac{4}{9}$$



Minnesota State High School Mathematics League

Individual Event

2003-04 Event 5C

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

- 512 1. In the binomial expansion

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

the sum of the coefficients is $1 + 4 + 6 + 4 + 1 = 16$. What is the sum of the coefficients in the expansion of $(a+b)^9$?

5
162 or 0.0309

2. In the game of Yahtzee, five dice are rolled. A consecutive sequence (i.e. 12345 or 23456) is called a *large straight*. What is the probability of getting a large straight in a single throw of the five dice?

- 3 3. The set S consists of four points placed in the plane so that no three are collinear. Through each pair of points, a line is drawn. When two of these lines intersect in a point $p \notin S$, p is called a *new point*. At most, how many new points can be created?

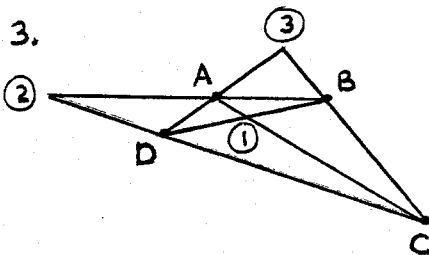
- 14 4. In how many ways can you add six odd natural numbers (i.e. positive integers) together to get 20? Numbers may be repeated, but changes in order are not to be counted as new solutions. That is,

$$9 + 3 + 3 + 1 + 1 + 3 \text{ and } 3 + 1 + 9 + 1 + 3 + 3$$

are regarded to be the same solution.

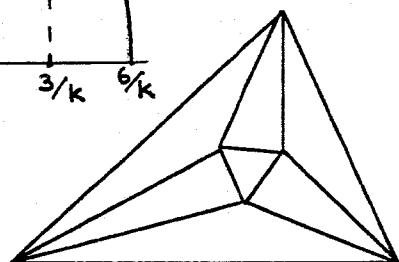
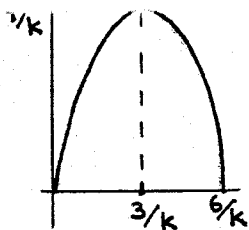
1. Set $a = b = 1$
 $(1+1)^9 = 2^9 = 512$

2. Think of obtaining 12345 by rolling one die five times. The first roll may come up in any of 5 ways, the second 4 ways, etc. Ways to get 12345: $5!$. Similarly, there are $5!$ ways to get 23456.
prob = $\frac{2 \cdot 5!}{6^5} = \frac{5}{162}$



4. The trick to counting is to have an organized way to list them — as at the right. There are 14 ways to get 20.

$$\begin{aligned} 15 + 1 + 1 + 1 + 1 + 1 &= 20 \\ 13 + 3 + 1 + 1 + 1 + 1 &= 20 \\ 11 + 5 + 1 + 1 + 1 + 1 &= 20 \\ 11 + 3 + 3 + 1 + 1 + 1 &= 20 \\ 9 + 7 + 1 + 1 + 1 + 1 &= 20 \\ 9 + 5 + 3 + 1 + 1 + 1 &= 20 \\ 9 + 3 + 3 + 3 + 1 + 1 &= 20 \\ 7 + 7 + 3 + 1 + 1 + 1 &= 20 \\ 7 + 5 + 5 + 1 + 1 + 1 &= 20 \\ 7 + 5 + 3 + 3 + 1 + 1 &= 20 \\ 7 + 3 + 3 + 3 + 3 + 1 &= 20 \\ 5 + 5 + 5 + 3 + 1 + 1 &= 20 \\ 5 + 5 + 3 + 3 + 3 + 1 &= 20 \\ 5 + 3 + 3 + 3 + 3 + 3 &= 20 \end{aligned}$$



Minnesota State High School Mathematics League

Individual Event

2003-04 Event 5D

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 20 minutes for this event.

- 6011 1. What will be the result if the sum of the first 2003 odd counting numbers (that is the first 2003 odd positive integers) is subtracted from the sum of the first 2004 even counting numbers?

- 5.5 mph 2. Mary runs each morning following a park trail that takes her in 9 minutes time to a little pond $\frac{3}{4}$ of a mile from her home. There she meets Jenny, and they take 16 minutes as they run together the 1.8 mile trail around the pond. A bit tired by then, Mary takes 11 minutes for the run back home. What is Mary's average rate in miles per hour for the entire trip?

- 2 3. Figure 3 shows a square inscribed in a circle of radius 2. One edge of the square is the diameter of a semicircle sitting on top of the square. The shaded area inside the semi-circle, outside of the larger circle is called a *lune*. Determine the area of the lune.

- 4 4. For what choice of $k > 0$ will $f(x) = \sqrt{6x - kx^2}$ have the same domain and range?

$$\begin{aligned} 2. \text{ distance} &= .75 + 1.8 + .75 = 3.3 \\ \text{time} &= \frac{9 + 16 + 11}{60} = \frac{36}{60} = \frac{3}{5} \\ \text{rate} &= \frac{33/10}{3/5} = \frac{33}{6} = \frac{11}{2} \end{aligned}$$

$$\begin{aligned} 3. \text{ Area } (L) &= A(L+G) + A(T) - A(G+T) \\ AB &= 2\sqrt{2}; A(L+G) = \frac{1}{2}\pi \left[\frac{2\sqrt{2}}{2}\right]^2 = \pi \\ A(T) &= \frac{1}{4}(2\sqrt{2})^2 = 2 \\ A(G+T) &= \frac{1}{4}\pi(2)^2 = \pi \\ A(L) &= \pi + 2 - \pi = 2 \end{aligned}$$

$$\begin{aligned} 4. \text{ Let } g(x) &= 6x - kx^2 = -k\left(x^2 - \frac{6}{k}x + \frac{9}{k^2}\right) + \frac{9}{k} \\ \text{The graph of } y &= g(x) \text{ is the parabola } y - \frac{9}{k} = -k\left(x - \frac{3}{k}\right)^2 \text{ (top of page)} \\ f(x) &= \sqrt{g(x)} \text{ has domain } [0, \frac{6}{k}] \text{ and range } [0, \sqrt{\frac{9}{k}}]. \\ \text{If } \frac{6}{k} &= \frac{3}{\sqrt{k}}, \text{ then } k = 4 \end{aligned}$$

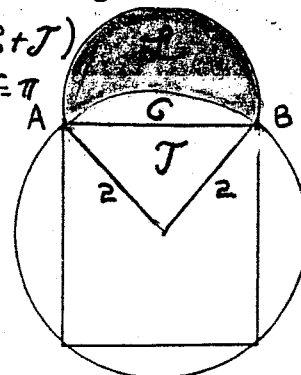
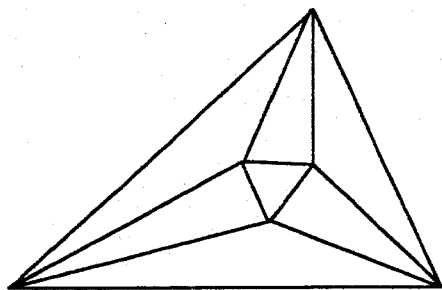


Figure 3

$$\begin{aligned} 1. 2[1 + 2 + \dots + 2002 + 2003 + 2004] &= 2 + 4 + \dots + 4004 + 4006 + 4008 \\ [2(1)-1] + [2(2)-1] + \dots + [2(2002)-1] + [2(2003)-1] &= 1 + 3 + \dots + 4003 + 4005 \\ \underbrace{1 + 1 + \dots + 1 + 1}_{2003 \text{ ones}} + 4008 &= 6011 \end{aligned}$$



Minnesota State High School Mathematics League

Team Event

2003-04 Meet 5

Each question is worth 4 points. Team members may cooperate in any way, but at the end of twenty minutes, one set of answers is to be submitted. Put answers on the lines provided.

15

1. The set S consists of five points placed in the plane, no three being collinear. A line is drawn through each pair of points. When two of these lines intersect in a point $p \notin S$, then p is called a *new point*. At most, how many new points can be created?

(E) 61727

2. Partition the numbers in the set $\{1, 2, 3, \dots, 123456\}$ into two sets as follows. In set E , place all the integers whose digits add to an even number (so, for example, 4729 goes in E); and in O place all the integers whose digits add to an odd number (so 36967 goes in O). How many numbers get placed in E ? in O ?

(O) 61729

346

3. Twelve railroad cars, four of them boxcars, four flatcars, and four tankcars, are to be coupled together. In how many ways can this be done if there are to be no boxcars in the first four cars of the train, no flatcars in the middle four cars of the train, and no tankcars in the last four cars?

R - r

4. Let A, B, C , and D be four points on a line (Figure 4) such that $AB = CD$. Semi-circles are drawn above the line with diameters $AB = CD = 2r$ and $AD = 2R$, and another semi-circle with diameter BC is drawn below the line. The shaded area bounded by these four semi-circles is called a *salinon*. Its area is equal to the area of a circle of radius x . Express x in terms of R and r .

5. Let the line L through $A(-1,0)$ and $B(0,t)$ intersect the unit circle at $P(x,y)$. Express x and y in terms of t .

$$x = \frac{1-t^2}{1+t^2}$$

$$y = \frac{2t}{1+t^2}$$

5/2

6. The long leg of a 5-12-13 right $\triangle ABC$ is used as the diameter of a semi-circle that cuts the hypotenuse at D (Figure 6). A line tangent to the circle at D cuts AC at E . What is the length of CE ?

6. $\angle ACD = \angle ABC = \alpha$
 (sides are \perp)
 $EC = ED$ so $\triangle CDE$ is isos.
 $\therefore \angle CDE = \alpha$
 $\angle CDB = 90^\circ$ (inscribed in a semicircle)
 Let $\beta = \angle BCD = \angle EDA = \angle CAD$
 (all complements of α)
 $\therefore \triangle ADE$ is isosceles, so
 $ED = EA$. Thus $EC = EA$
 $CE = 5/2$

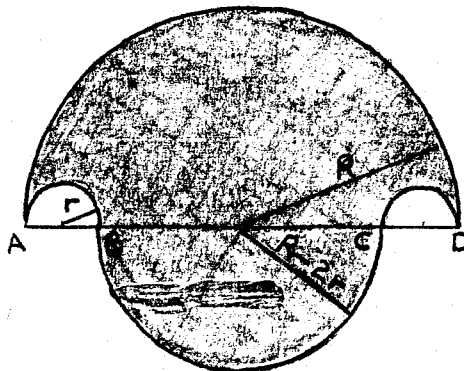


Figure 4

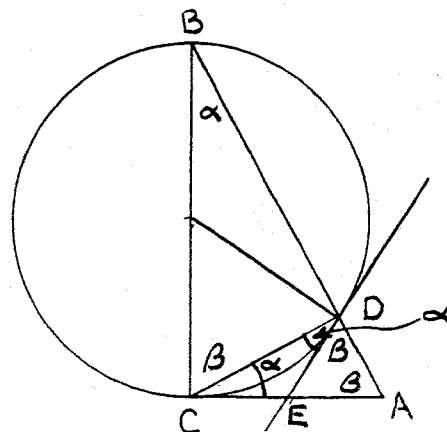


Figure 6

Solutions for Team Event 5

2. Consider any ten successive integers that begin with a multiple of ten. Examples:

Even	Odd	Even	Odd	Even	Odd
0	1	31	32	1270	1271
2	3	33	34	1272	1273
4	5	35	36	1274	1275
6	7	37	38	1276	1277
8	9	39		1278	1279

Thus, E and O will have the same number of integers after placing of 0, ---, 123449. Then we place

Even	Odd
	123450
123451	123452
123453	123454
123455	123456

∴ O has one more entry than E; but exclude the first 0.

3. Let the first four cars include k flatcars, $0 \leq k \leq 4$, and $4-k$ tank cars.

Since there are no tank cars in the last section, there must be k tank cars in the middle section, together with $4-k$ boxcars. Finally, the last section must contain k boxcars, $4-k$ flatcars. Thus, the train's make up is completely determined once we have placed k flatcars in the first section, k tank cars in the middle section, and k boxcars in the last section.

For a choice of k , $0 \leq k \leq 4$, there are $\binom{4}{k}$ placements in each section. Thus, for each k , there are $\binom{4}{k}^3$ arrangements. The total number is therefore $\binom{4}{0}^3 + \binom{4}{1}^3 + \binom{4}{2}^3 + \binom{4}{3}^3 + \binom{4}{4}^3 = 346$

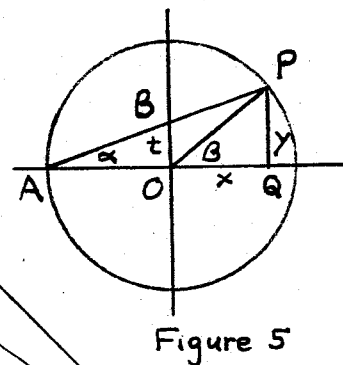
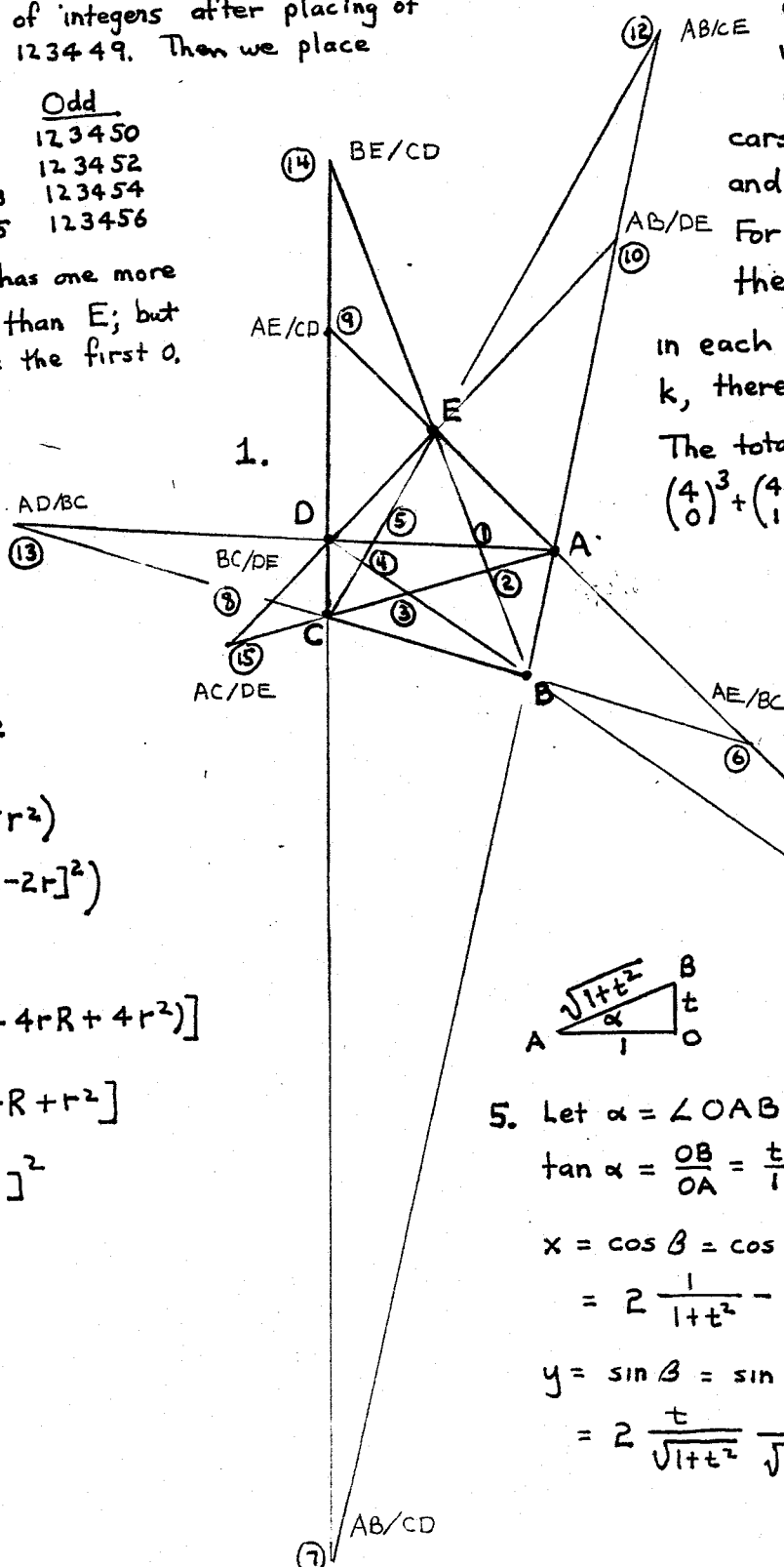
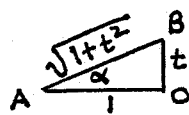


Figure 5

$$\begin{aligned}
 4. & \frac{1}{2} \pi R^2 \\
 & - 2 \left(\frac{1}{2} \pi r^2 \right) \\
 & + \frac{1}{2} (\pi [R-2r]^2) \\
 = & \pi \left[\frac{1}{2} R^2 - r^2 + \frac{1}{2} (R^2 - 4rR + 4r^2) \right] \\
 = & \pi [R^2 - 2rR + r^2] \\
 = & \pi [R-r]^2 \\
 x = & R-r
 \end{aligned}$$



5. Let $\alpha = \angle OAB$; $\beta = \angle POQ = 2\alpha$
- $$\tan \alpha = \frac{OB}{OA} = \frac{t}{1} = \frac{y}{1+x}$$
- $$x = \cos \beta = \cos 2\alpha = 2 \cos^2 \alpha - 1$$
- $$= 2 \frac{1}{1+t^2} - \frac{1+t^2}{1+t^2} = \frac{1-t^2}{1+t^2}$$
- $$y = \sin \beta = \sin 2\alpha = 2 \sin \alpha \cos \alpha$$
- $$= 2 \frac{t}{\sqrt{1+t^2}} \frac{1}{\sqrt{1+t^2}} = \frac{2t}{1+t^2}$$